Solitons in a class of systems of two coupled real scalar fields

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This paper deals with systems of two coupled real scalar fields in bidimensional spacetime. We show that when the potential that specifies the system presents a particular form, we are led to first-order equations that solve the second-order equations of motion for static fields. The first-order equations can be seen as a dynamical system, and the static classical solutions present minimum energy, and are classically stable. We consider explicit examples to illustrate the general procedure. In particular, we introduce a specific system that can be used to model ferroelectric crystals. [S1063-651X(96)01009-4]

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I. INTRODUCTION

This paper deals with solitons in nonlinear systems of coupled real scalar fields. This is important in physics since there are many systems that are described via nonlinear coupling of their relevant degrees of freedom, and usually lead to defect or soliton solutions. In the standard route to solitons one starts by examining a specific system, and after working out the relevant physical contents, one gets to a set of coupled nonlinear differential equations. Due to the basic principles envoked to extract the physical contents, the equations one gets are in general second-order differential equations.

In this paper we shall follow a different approach, e.g., we shall work to implement a method to solve sets of coupled nonlinear differential equations. In this way, to get to specific nonlinear systems one is just left to the work of mapping sets of coupled nonlinear differential equations to the particular systems one is interested in. To implement this idea we shall take advantage of a method that is well known to particle physicists. In this case we start by writing a Lagrangian density corresponding to some set of real scalar fields in bidimensional spacetime. The motivation from field theory is evident, and we believe we can get from this picture to applications in condensed matter systems. This idea originated from the investigations already presented in some recent works [1,2], and here we want to go further to generalize the method introduced there.

The investigation will be done by examining systems of coupled real scalar fields in bidimensional spacetime. For simplicity, we consider the case of two fields, namely, ϕ and χ. Systems of this kind are usually described by a Lagrangian density that contains a potential $U=U(\phi,\chi)$, in general a nonlinear function of the real scalar fields. It is this nonlinearity that enlarges the scope of the problem, since it can be mapped to many interesting systems in nonlinear science.

In field theory, systems of coupled scalar fields have intrinsec interest [3], and can be easily extended to incorporate a very interesting idea [4] in which topological solitons have internal structure. This idea was explored in another paper [5], in which we considered the presence of topological defects inside domain walls. In other branches of physics there may also be applications, and this we shall further explore in the present paper.

This paper is organized as follows. In Sec. II we introduce a Lagrangian density, which represents a general class of systems, to develop a method of searching for soliton solutions. In Sec. III we present a proof of classical or linear stability of solitons that one can find in this class of systems. To illustrate the general procedure, in Sec. IV we investigate specific systems, in order to consider explicit applications. In Sec. V, we present some comments and conclusions.

II. A CLASS OF SYSTEMS OF COUPLED FIELDS

A general Lagrangian density describing a relativistic system of two coupled real scalar fields in bidimensional spacetime is given by

$$\mathcal{L}=\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi+\frac{1}{2}\partial_\alpha\chi\partial^\alpha\chi-U(\phi,\chi), \tag{1}$$

where $U=U(\phi,\chi)$ is the potential, which specifies the particular system one is interested in. Our notation is common: we are using natural units, in which $\hbar=c=1$, and the metric tensor $g^{\alpha\beta}$ is diagonal, with $g^{00}=-g^{11}=1$. This system leads to the following set of equations of motion:

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\frac{\partial U}{\partial\phi}=0 \tag{2}$$

and

$$\frac{\partial^2\chi}{\partial t^2}-\frac{\partial^2\chi}{\partial x^2}+\frac{\partial U}{\partial\chi}=0. \tag{3}$$

In the standard way of searching for soliton solutions one considers static field configurations, and so $\phi=\phi(x)$ and $\chi=\chi(x)$. In this case, the equations of motion become

$$\frac{d^2\phi}{dx^2}=\frac{\partial U}{\partial\phi}, \tag{4}$$

and

$$\frac{d^2\chi}{dx^2}=\frac{\partial U}{\partial \chi}. \tag{5}$$

In the above system, the potential $U(\phi,\chi)$ in general is a nonlinear function of the fields, and so the equations of motion (4) and (5) constitute a nonlinear system of coupled second-order differential equations. Here we face a mathematical difficulty and it seems to be interesting to introduce a method to avoid this intrinsic barrier. We think we can make an important step toward circumventing this problem. To do this, however, we must constrain the potential in some specific way [1,2], as we are going to show, and this will certainly restrict our investigation. This is the price one has to pay, although we get a large class of systems, which can be investigated in a simpler way.

As stated in the Introduction, we shall go further to generalize our former investigations [1,2]. To do this, we consider the potential in the form

$$U(\phi,\chi)=\tfrac{1}{2}V^2(\phi,\chi)+\tfrac{1}{2}W^2(\phi,\chi), \tag{6}$$

in which the functions $V(\phi,\chi)$ and $W(\phi,\chi)$ are in principle arbitrary but continuous twice differentiable functions of the fields ϕ and χ. In this case, the equations of motion describing static field configurations become

$$\frac{d^2\phi}{dx^2}=V\frac{\partial V}{\partial \phi}+W\frac{\partial W}{\partial \phi} \tag{7}$$

and

$$\frac{d^2\chi}{dx^2}=V\frac{\partial V}{\partial \chi}+W\frac{\partial W}{\partial \chi}. \tag{8}$$

This does not seem to give a good answer to the above mentioned problem, but this is not so, as we are now going to show.

Here we follow the procedure introduced in [1,2]—see also Ref. [6]. In this case one investigates the energy corresponding to static field configurations. For the system given by (1) and (6) the energy-momentum tensor can be easily obtained: its explicit components are, for static field configurations,

$$T_{00}=\frac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\chi}{dx}\right)^2+V^2+W^2\right\} \tag{9}$$

and

$$T_{11}=\frac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\chi}{dx}\right)^2-V^2-W^2\right\}, \tag{10}$$

and $T_{01}=T_{10}=0$. We use (9) to write the energy as

$$E=\frac{1}{2}\int_{-\infty}^{\infty}dx\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\chi}{dx}\right)^2+V^2+W^2\right\}. \tag{11}$$

This expression for the energy can be rewritten in the form $E=E'+E''$, where E' is given by

$$E'=\frac{1}{2}\int_{-\infty}^{\infty}dx\left\{\left(\frac{d\phi}{dx}-V\right)^2+\left(\frac{d\chi}{dx}-W\right)^2\right\}, \tag{12}$$

and E'' reads

$$E''=\int_{-\infty}^{\infty}dx\left\{V\frac{d\phi}{dx}+W\frac{d\chi}{dx}\right\}. \tag{13}$$

We now introduce a general function $H=H(\phi,\chi)$ to write a new quantity E_M in the following form:

$$E_M=\int_{-\infty}^{\infty}dx\,\frac{dH}{dx}=H(\phi(\infty),\chi(\infty))-H(\phi(-\infty),\chi(-\infty)). \tag{14}$$

We use the chain rule to get

$$\frac{dH}{dx}=\frac{\partial H}{\partial \phi}\frac{d\phi}{dx}+\frac{\partial H}{\partial \chi}\frac{d\chi}{dx}. \tag{15}$$

Then, if we introduce the conditions

$$\frac{\partial H}{\partial \phi}=V,\quad \frac{\partial H}{\partial \chi}=W, \tag{16}$$

we obtain $E''=E_M$ in an obvious way. Moreover, from (12), (13), and (16) we recognize that E_M is the minimum value for the energy, which is achieved when we impose the conditions

$$\frac{d\phi}{dx}=V \tag{17}$$

and

$$\frac{d\chi}{dx}=W, \tag{18}$$

since in this case the contribution E' given in (12) is zero. Here we also notice that $T_{11}=0$, as shown by (10).

The above equations (17) and (18) are first-order equations, and we can use them to verify that

$$\frac{d^2\phi}{dx^2}=V\frac{\partial V}{\partial \phi}+W\frac{\partial V}{\partial \chi} \tag{19}$$

and

$$\frac{d^2\chi}{dx^2}=V\frac{\partial W}{\partial \phi}+W\frac{\partial W}{\partial \chi}. \tag{20}$$

We now compare (7) to (19), and (8) to (20), to easily see that we can make the first-order equations (17) and (18) to solve the second-order equations of motion (7) and (8), when we impose the condition

$$\frac{\partial V}{\partial \chi}=\frac{\partial W}{\partial \phi}. \tag{21}$$

Here we note that the above condition (21) is just the condition for the existence of a continuously twice differentiable function $H=H(\phi,\chi)$ satisfying (16). Therefore, for the general system (1), when the potential has the specific form (6), the second-order differential equations of motion (7) and (8) are solved by the first-order differential equations (17) and (18), if one imposes condition (21).

We believe that this is an important step since now instead of working with the second-order equations of motion, we have to deal with first-order differential equations. In this case we can think of the set of first-order equations as describing a dynamical system, and so we can use all the mathematical tools available to dynamical systems to deal with it. Furthermore, the energy corresponding to static field configurations is bounded from below, and gets to its minimum value given by (14), where the function $H(\phi,\chi)$ can be obtained from conditions (16). Here we recognize that each function $H(\phi,\chi)$ defines a specific system, and so we have a general class of systems of coupled real scalar fields.

This same function $H(\phi,\chi)$ can be used to define topological sectors. Here we follow [7] and introduce the topological current

$$J_T^{\alpha}=\epsilon^{\alpha\beta}\partial_{\beta}H(\phi,\chi), \tag{22}$$

which is trivially conserved, thanks to the asymmetry of the Levi-Civita tensor: $\epsilon^{01}=-\epsilon^{10}=1$ and $\epsilon^{00}=\epsilon^{11}=0$. The corresponding topological charge is given by

$$Q_T=\int_{-\infty}^{\infty}dx J_T^0=H(\phi(\infty),\chi(\infty))-H(\phi(-\infty),\chi(-\infty)). \tag{23}$$

In this case the topological charge is equal to the energy of the static field configurations. The vacuum sector, which is identified by time and space independent field configurations, has zero topological charge. We use the topological charge, which is conserved, to introduce topological sectors: different nonvanishing topological charges define different topological sectors. We then have a general classification scheme for the topological solutions.

The issue concerning topological solitons will be further considered in the next section, where we investigate explicit examples. Before working with explicit systems, however, let us show that soliton solutions in systems of the above general class are classically stable.

III. CLASSICAL STABILITY

In this section we deal with classical or linear stability [2] of soliton solutions one can find in the general class of systems considered in Sec. II. To investigate classical or linear stability [8], firstly we consider $\phi=\phi(x)$ and $\chi=\chi(x)$ as a pair of solutions to the first-order equations (17) and (18). We then write

$$\phi(x,t)=\phi(x)+\eta(x,t) \tag{24}$$

and

$$\chi(x,t)=\chi(x)+\zeta(x,t), \tag{25}$$

where $\eta(x,t)$ and $\zeta(x,t)$ are fluctuations about the classical solutions. We now substitute the above $\phi(t,x)$ and $\chi(t,x)$ into the equations of motion (2) and (3) to get, up to first-order in the fluctuations,

$$S_2\begin{pmatrix}\eta\\ \zeta\end{pmatrix}=-\frac{\partial^2}{\partial t^2}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \tag{26}$$

where S_2 is a second-order Schrödinger-like differential operator, given by

$$S_2=-\frac{d^2}{dx^2}+\begin{pmatrix}U_{\phi\phi} & U_{\chi\phi}\\ U_{\phi\chi} & U_{\chi\chi}\end{pmatrix}, \tag{27}$$

where we are using the notation $U_{\phi\phi}=\partial^2U/\partial\phi^2$, $U_{\chi\phi}=\partial^2U/\partial\chi\partial\phi$, and so on. By assumption, the classical solutions are time independent, and so we can separate space and time in (26). In the case restricted to single harmonics, for instance, we can write $\eta(x,t)=\eta(x)\cos(wt)$ and $\zeta(x,t)=\zeta(x)\cos(wt)$. In this case we have to change $-(\partial^2/\partial t^2)\to w^2$ in (26).

We focus our attention on the time-independent operator S_2, and now we can write, for $U(\phi,\chi)$ given by (6),

$$U_{\phi\phi}=VV_{\phi\phi}+V_{\phi}^2+WW_{\phi\phi}+W_{\phi}^2, \tag{28}$$

$$U_{\phi\chi}=V_{\phi}V_{\chi}+VV_{\phi\chi}+W_{\phi}W_{\chi}+WW_{\phi\chi}, \tag{29}$$

$$U_{\chi\chi}=VV_{\chi\chi}+V_{\chi}^2+WW_{\chi\chi}+W_{\chi}^2, \tag{30}$$

and

$$U_{\chi\phi}=V_{\phi}V_{\chi}+VV_{\chi\phi}+W_{\phi}W_{\chi}+WW_{\chi\phi}. \tag{31}$$

Here we notice that the second-order differential operator S_2 is self-adjoint since by Schwarz's theorem we have

$$V_{\phi\chi}=V_{\chi\phi}, \quad W_{\phi\chi}=W_{\chi\phi}. \tag{32}$$

However, for classical stability to apply we must show that S_2 is positive semidefinite [8].

We follow this reasoning, and we use the first-order differential equations (17) and (18) to introduce the operators

$$S_1^{\pm}=\pm\frac{d}{dx}+\begin{pmatrix}V_{\phi} & V_{\chi}\\ W_{\phi} & W_{\chi}\end{pmatrix}. \tag{33}$$

We now recall condition (21) to see that the above operators $S_1^{\pm}$ are adjoint of each other. Moreover, the very condition $V_{\chi}=W_{\phi}$ allows one to write the second-order differential operator S_2 in terms of the first-order differential operators $S_1^{\pm}$ in the following form:

$$S_2=S_1^+S_1^-. \tag{34}$$

We then conclude that the second-order differential operator S_2 is self-adjoint and non-negative, and this fact assures classical or linear stability of all soliton solutions the general class of systems we have introduced in the former section can comprise.

From the above investigation we can rather naturally introduce another second-order differential operator $\overline{S_2}$ given by

$$\overline{S_2}=S_1^-S_1^+. \tag{35}$$

Evidently, this new operator is also self-adjoint and non-negative. Moreover, the spectrum of $\overline{S_2}$ is equal to the spectrum of S_2, except for the zero eigenvalue of S_2. This is so because the equation

$$S_1^- S_1^+ S_1^- = S_1^- S_2 = \overline{S_2} S_1^- \tag{36}$$

is always true, unless S_1^- presents a null eingenvector, to give the zero eigenvalue for S_2. We notice that this is indeed the case here, since the fluctuations $\eta(x)=d\phi/dx=V$ and $\zeta(x)=d\chi/dx=W$ constitute the null eigenvector of S_1^-. This result is just a consequence of the fact that the class of systems we are investigating engenders translational invariance.

IV. APPLICATIONS

The investigations introduced in the former sections are done on general grounds, and can be generalized to the case of three or more fields straightforwardly. In the present section we shall investigate some examples, in order to illustrate the general procedure. In Sec. IV A we investigate a system with the motivation of showing how one can map systems of coupled scalar fields in macroscopic chains. In Sec. IV B we introduce another system to illustrate the improved version of our method, as developed in Sec. II.

A. Interactions up to the fourth power

In Ref. [1] we investigated several systems, and presented explicit soliton solutions. As a particularly interesting system introduced there, let us write

$$V_1(\phi,\chi)=\lambda(\phi^2-a^2)+\tfrac{1}{2}\mu\chi^2 \tag{37}$$

and

$$W_1(\phi,\chi)=\mu\phi\chi. \tag{38}$$

The potential that specifies the above system is given by

$$U_1(\phi,\chi)=\tfrac{1}{2}\lambda^2(\phi^2-a^2)^2+\tfrac{1}{2}\lambda\mu(\phi^2-a^2)\chi^2+\tfrac{1}{8}\mu^2\chi^4 + \tfrac{1}{2}\mu^2\phi^2\chi^2 \tag{39}$$

and the function $H(\phi,\chi)$ introduced in (16) is written as

$$H_1(\phi,\chi)=\lambda\phi(\tfrac{1}{3}\phi^2-a^2)+\tfrac{1}{2}\mu\phi\chi^2. \tag{40}$$

Here we notice that when $\mu=0$, the two fields decouple and the ϕ field presents two minimum states at $\phi^2=a^2$. This system was already investigated in [1] and there are soliton solutions. Some of the soliton solutions are the pair $\chi=0$ and

$$\phi(x)=a\tanh(\lambda a x), \tag{41}$$

and also the pair

$$\phi(x)=a\tanh(\mu a x) \tag{42}$$

and

$$\chi(x)=\pm a[2(\lambda/\mu-1)]^{1/2}\operatorname{sech}(\mu a x), \tag{43}$$

with $\lambda/\mu>1$.

The above system can describe the continuum version of the following macroscopic chain: let us have a system represented by $\cdots X \cdots X \cdots$, where $\cdots$ stand for the binding between atoms or group of atoms X. We suppose that X has an internal degree of freedom that can assume two distinct minimum values. We represent such a system as

$$\cdots\nearrow\cdots\nearrow\cdots\nearrow\cdots\nearrow\cdots\nearrow\cdots$$

$$\cdots\nwarrow\cdots\nwarrow\cdots\nwarrow\cdots\nwarrow\cdots\nwarrow\cdots$$

and these are the two possible (degenerate) minimum energy states of the system. In this representation, arrows refer to the internal degree of freedom of X, which can be described by the angle between the arrow and the direction perpendicular to the chain. We map this degree of freedom to the scalar field ϕ of our system, and the two minimum energy states are described by $\phi^2=a^2$.

We now freeze the degree of freedom for motion parallel to the chain. In this case we can map the field χ to the degree of freedom that allows deviation of X from its equilibrium position, in the direction perpendicular to the chain. In this case, a defect in this chain can be roughly represented by

$$\cdots\nwarrow\cdots\nwarrow\cdots\uparrow\cdots\nearrow\cdots\nearrow\cdots.$$

For simplicity, here we are drawing a defect in the absence of interaction between the two degrees of freedom, with $\mu=0$. This situation is represented by the pair of solutions $\chi=0$ and ϕ given by (41). Of course, when interaction between ϕ and χ is turned on, we have $\mu\neq0$, and the vertical positions of X deviate from the equilibrium position, as one can easily realize. Such a situation is described by the pair of solutions given by (42) and (43).

Here we notice that the system of coupled real scalar fields defined by (37) and (38) can be used to map the continuum version of the above model of macroscopic chain. This result is very interesting because this specific macroscopic chain can be used to model ferroelectric crystals [9]. In this case the internal degree of freedom of X represents the microscopic electric dipole associated with a molecular group such as the $NaNO_2$ group.

B. Interactions up to the sixth power

In the present subsection we shall investigate another explicit model, to illustrate the general procedure. Our main point here is to present an investigation that follows the improved version of our method, as developed in Sec. II. With this specific motivation, let us now choose V and W in the following form:

$$V_2(\phi,\chi)=\phi(\lambda\phi^2+\mu\chi^2-\lambda a^2) \tag{44}$$

and

$$W_2(\phi,\chi)=\chi(\lambda\chi^2+\mu\phi^2-\lambda a^2). \tag{45}$$

For simplicity we consider a real, and λ and μ real and positive parameters, with $0<\mu<\lambda$. The potential that specifies the above system is given by

$$U_2(\phi,\chi)=\tfrac{1}{2}\phi^2(\lambda\phi^2+\mu\chi^2-\lambda a^2)^2 + \tfrac{1}{2}\chi^2(\lambda\chi^2+\mu\phi^2-\lambda a^2)^2 \tag{46}$$

and the function $H(\phi,\chi)$ introduced in (16) is written as

$$H_2(\phi,\chi)=\tfrac{1}{2}\lambda\phi^2(\tfrac{1}{2}\phi^2-a^2)+\tfrac{1}{2}\lambda\chi^2(\tfrac{1}{2}\chi^2-a^2)+\tfrac{1}{2}\mu\phi^2\chi^2. \quad (47)$$

The above choice of parameters is one among several possibilities, and makes the system to present the Z_4 symmetry, for $0<\mu<\lambda$. Evidently, other choices are allowed, but we have chosen the above one to introduce an illustrative example of the general procedure. By the way, here we note that the limit $\mu\to 0$ decouples the two fields, and the limit $\mu\to\lambda$ introduces a continuum symmetry, namely the U(1) symmetry.

We now use (44) and (45) to write the first-order equations (17) and (18) in the form

$$\frac{d\phi}{dx}=\phi(\lambda\phi^2+\mu\chi^2-\lambda a^2) \quad (48)$$

and

$$\frac{d\chi}{dx}=\chi(\lambda\chi^2+\mu\phi^2-\lambda a^2). \quad (49)$$

We deal with this system of first-order differential equations as a dynamical system. The singular points are the zeros of V and W, and so they are the minima of the potential. There are nine singular points, namely, (0,0), $(\pm a,0)$, $(0,\pm a)$, and $(\pm a/\sqrt{1+\mu/\lambda},\pm a/\sqrt{1+\mu/\lambda})$. The last four points are obtained from the equations

$$\phi^2+\frac{\mu}{\lambda}\chi^2=a^2$$

and

$$\chi^2+\frac{\mu}{\lambda}\phi^2=a^2,$$

which are equations of ellipses in the (ϕ,χ) plane. The solutions are the four intersection points of the two ellipses, which lie on the lines $\phi^2=\chi^2$. We note that in the limit $\mu\to\lambda$ these two ellipses collapse into a single circle, and the solutions are now the whole circle, thus indicating the presence of the continuum U(1) symmetry, as stated in the former paragraph.

The nine singular points in this system are classified as stable (0,0), unstable $(\pm a/\sqrt{1+\mu/\lambda},\pm a/\sqrt{1+\mu/\lambda})$, and saddle $[(\pm a,0),(0,\pm a)]$ points. In accordance with the procedure we have introduced at the very end of Sec. II, here we can identify some topological sectors. The first sector, sector 1, is the sector that connects the points (0,0) and $(a,0)$. It topological charge $(Q_T=E)$ is given by

$$Q_T^1=E_1=\tfrac{1}{4}\lambda a^4. \quad (50)$$

The second sector, sector 2, connects (0,0) to $(a/\sqrt{1+\mu/\lambda})$. It has the following topological charge

$$Q_T^2=E_2=\frac{1}{4}\lambda a^4\left(\frac{2}{1+\dfrac{\mu}{\lambda}}\right). \quad (51)$$

We note that $E_1<E_2$, and the limit $\mu\to\lambda$ makes $E_2\to E_1$. In this case these two sectors collapse into a single one, and this is a consequence of the fact that in the limit $\mu\to\lambda$ we are lead to the continuum U(1) symmetry.

To find explicit soliton solutions we consider each one of the above two sectors separately. The simplest case is sector 1, which connects $(a,0)$ to (0,0). In this case we set $\chi=0$ to get

$$\frac{d\phi}{dx}=\lambda\phi(\phi^2-a^2). \quad (52)$$

and so the ϕ field supports ϕ^6 kink solutions [7]. The explicit solutions are $\chi=0$ and

$$\phi(x)=2^{-1/2}a[1-\tanh\lambda a^2(x+\overline{x})]^{1/2}, \quad (53)$$

where $\overline{x}$ is an arbitrary point, which identifies the ''center'' of the solution. Next, in sector 2, which connects $(a/\sqrt{1+\mu/\lambda},a/\sqrt{1+\mu/\lambda})$ to (0,0), we set $\chi=\phi$ to get

$$\frac{d\phi}{dx}=\lambda\phi[(1+\mu/\lambda)\phi^2-a^2]. \quad (54)$$

In this case the explicit solutions are

$$\phi(x)=\chi(x)=\frac{2^{-1/2}a}{\sqrt{1+\mu/\lambda}}[1-\tanh\lambda a^2(x+\overline{x})]^{1/2}. \quad (55)$$

We can identify another topological sector, connecting the points $(a,0)$ and $(a/\sqrt{1+\mu/\lambda},a/\sqrt{1+\mu/\lambda})$. In this case the corresponding topological charge is

$$Q_T^3=E_3=\frac{1}{4}\lambda a^4\left(\frac{1-\mu/\lambda}{1+\mu/\lambda}\right), \quad (56)$$

and so it is the lower charge available, satisfying $0<E_3<E_1$. Here we note that $E_3\to 0$ when $\mu\to\lambda$, which is the limit that leads to the continuum U(1) symmetry. This result is in agreement with the Goldstone theorem, which states that no energy is spent to link two points on the circle, for $\mu=\lambda$. We have being unable to find explicit analytical solutions in this third sector.

To introduce another model, let us write the following new functions

$$V_3=\chi(\lambda\chi^2+3\mu\phi^2+\nu) \quad (57)$$

and

$$W_3=\phi(\mu\phi^2+3\lambda\chi^2+\nu), \quad (58)$$

where λ, μ , and ν are real parameters. In this case we are led to another system, specified by

$$\frac{d\phi}{dx}=\chi(\lambda\chi^2+3\mu\phi^2+\nu) \quad (59)$$

and

$$\frac{d\chi}{dx}=\phi(\mu\phi^2+3\lambda\chi^2+\nu). \quad (60)$$

The investigation of this new system follows the one we have just done, and so we omit it here.

We notice, however, that the above system of equations very much resembles the first-order equations recently proposed to model systems of diatomic chains [10,11]. To be explicit, we recall that the first-order equations presented in [10] are given by, using our notation,

$$\frac{d\phi}{dx}=\chi(\chi^2+3\phi^2+a) \tag{61}$$

and

$$\frac{d\chi}{dx}=-\phi(\phi^2+3\chi^2+b), \tag{62}$$

where $a=1-w/w_1$ and $b=1-w/w_2$, with frequencies defined in [10]. In this case one can easily check that the above system of equations cannot be introduced by any function $H=H(\phi,\chi)$, and so it is not of the class we are investigating in here. This is also interesting, at least to show that the investigation of classical stability of the solutions found in [10] has to follow another route, not the one we have introduced in Sec. III.

V. COMMENTS AND CONCLUSIONS

To summarize we recall that we have investigated a general class of systems of coupled real scalar fields. This class of systems is defined by

$$\mathcal{L}=\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi+\frac{1}{2}\partial_\alpha\chi\partial^\alpha\chi-\frac{1}{2}\left(\frac{\partial H}{\partial\phi}\right)^2-\frac{1}{2}\left(\frac{\partial H}{\partial\chi}\right)^2,$$

where $H=H(\phi,\chi)$ is a smooth function of the fields ϕ and χ. In this case, the second-order equations of motion corresponding to static field configurations are solved by the following first-order equations

$$\frac{d\phi}{dx}=\frac{\partial H}{\partial\phi}$$

and

$$\frac{d\chi}{dx}=\frac{\partial H}{\partial\chi}.$$

The above set of first-order differential equations can be seen as a dynamical system, and so we can take advantage of all the mathematical tools available to dynamical systems to deal with it. We have also shown that the energy corresponding to static configurations in this class of systems gets to its lower bound, and that all soliton solutions the system can comprise are classically stable. This is interesting since solitons we can find in systems belonging to this class will certainly play some important role in understanding general properties of the respective systems.

In Sec. IV we have investigated some specific systems. The first one, which contains interactions up to the fourth power in the coupled fields, was already introduced in [1]. Here we have shown how to map that system in a macroscopic chain that can model ferroelectric crystals. The second example contains interactions up to the sixth power, and was introduced with the motivation of showing how to deal with issues that naturally appear in the improved version of our method of searching for soliton solutions in systems of coupled fields, as presented in Sec. II.

Some of the soliton solutions we have found in the second example resemble the solutions introduced in [12], describing ionic and orientational defects in hydrogen-bonded networks, and the twiston found in [13] for polyethelene. In these works, however, the systems intestigated present periodic interactions among the relevant degrees of freedom considered there. Here we realize that a natural extension of our method concerns investigating systems that present periodic interactions between the two coupled scalar fields. This will certainly enlarge the scope of the method, since periodic interactions seem to be most appropriate to model periodic structures one usually finds in condensed matter, and in organic and biological systems. This and other related issues are presently under consideration.

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